

# Circular stationary cyclic symmetric spacetimes; conformal flatness

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A subclassification of stationary spacetimes, endowed with one timelike and one spacelike Killing vectors, i.e., Petrov  $G_2I$  on  $T_2$  spaces, is proposed. Special attention deserves the Collison's theorem [1] and the branch of metrics circularly cyclicly (axially) symmetric possessing additionally the conformal flatness property reported by García and Campuzano [2].

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Although the definition of a cyclicly symmetric spacetime was introduced by Carter in 1970 [3], the literature on the topic is rather scarce. It was only until recently that the concept of cyclic symmetry attracts the attention of researchers in Relativity: Barnes (2001, 2000) [4], Heusler (1996) [5], Mars and Senovilla (1993) [6], Bičák and Schmidt (1984) [7]. Cyclic symmetry emerges as a slight generalization of the concept of axial symmetry. Following Carter, a spacetime is called cyclicly symmetric if and only if the metric is invariant under the action  $\pi$  of the one-parameter cyclic group  $SO(2)$ . A cyclicly symmetric spacetime is called axisymmetric if the set of fixed points of  $\pi$  is not empty, this set is referred to as the axis of symmetry. A spacetime is called stationary and axisymmetric if it is both stationary and axisymmetric, and if the Killing fields generating the symmetries commute with each other, see Heusler (1996) [5].

On the other hand, the normal form of spacetime metrics admitting an Abelian group of motions  $G_2$  acting on  $V_2$  has been determined by Petrov [8] by means of the metric tensor of the class  $G_2I$ :  $g_{ij} = g_{ij}(x^1, x^2)$  endowed with a pair of commuting Killing vectors  $\xi \equiv \partial_3$ , and  $\eta \equiv \partial_4$ . The 2-surface of transitivity (group orbit) spanned by  $\xi$  and  $\eta$  is timelike (timelike group orbits  $T_2$ ) when the square of the simple bivector  $\xi_{[a}\eta_{b]}$  is negative (for signature +2), see Kramer et al. [9].

Stationary cyclicly symmetric spacetimes are those Petrov  $G_2I$  spaces that possess timelike  $\xi \equiv \partial_t$  and spacelike  $\eta \equiv \partial_\phi$  fields of commuting Killing vectors, such that the trajectories of  $\partial_\phi$  are closed curves, which are referred to as Petrov  $G_2I$  on  $T_2$  spaces. The Killing trajectories span timelike 2-surface  $T_2$ . Therefore, the metric tensor  $g_{ij} = g_{ij}(x^1, x^2)$ ,  $i, j = 1, \dots, 4$ , possesses, in general, ten components independent of the Killing coordinates  $t$  and  $\phi$ , which can be reduced to six by coordinate transformations.

Circular stationary cyclicly symmetric spacetimes are those ones, which besides stationarity and cyclic symmetry, exhibit the simultaneous reflection (inversion in the Chandrasekhar's terminology [10]) symmetry  $(t, \phi) \rightarrow (-t, -\phi)$ . This invariance yields to the vanishing of the metric components  $g_{1t} = g_{2t} = g_{1\phi} = g_{2\phi} = 0$ , and thus the spacetime splits into two 2-surfaces orthogonal one to the other. The existence of 2-surfaces orthogonal to the group orbits (orthogonally transitive group) imposes conditions (circularity conditions) on the Killing vectors, see [9]:  $\xi_{[a;b}\xi_c\eta_{d]} = 0 = \eta_{[a;b}\eta_c\xi_{d]}$ . Most of the exact solutions of physical relevance belong to its two sub-branches: circular stationary cyclicly symmetric metric (CSCM) and circular stationary axisymmetric metric (CSAM)

As subclasses of Petrov  $G_2I$  spacetimes one can distinguish: non-circular stationary cyclicly symmetric metric (N-CSCM) and non-circular stationary axisymmetric metric (N-CSAM); to our knowledge, there are no exact solutions belonging to these subclasses in the literature, thus the search for this kind of metric is open.

Summarizing, within the class of Petrov  $G_2I$  on  $T_2$  spaces one distinguishes the following subclasses of metrics:

CSCM: Circular stationary cyclicly symmetric metric,

CSAM: Circular stationary axisymmetric metric,

N-CSCM: Non-circular stationary cyclicly symmetric metric,

N-CSAM: Non-circular stationary axisymmetric metric.

The class of circular stationary axisymmetric spacetimes, CSAM, has been studied extensively through the Lewis-Papapetrou metric and its variations, see Ref. [9].

The literature on exact solutions belonging to circular stationary cyclicly symmetric metrics, CSCM, is rather scarce. Nevertheless, the general form of conformally flat circular stationary cyclicly symmetric spacetimes, i.e., a CSCM supplemented with the property of conformal flatness, recently has been reported by us in Eq. (51) of Ref.

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[2]; the explicit form of the line-element is

$$ds^2 = e^{-2G(x,y)} \left[ \frac{dx^2}{(C_0 + C_1 x)(x^2 + 1)} + (C_0 + C_1 x) dy^2 + (x^2 + 1) d\phi^2 - (dt + x d\phi)^2 \right].$$

It is easy to establish that this metric does not possess an axis of symmetry; accomplishing a  $GL(2, R)$  transformation on the Killingian variables,  $t$  and  $\phi$ , it occurs that there is no way to determine an axis of symmetry because of the vanishing of the determinant of the transformation along any assumed existing axis. The cyclic property of the metric above was pointed out by Barnes and Senovilla [11], to whom we acknowledge their contribution to clarify this point (see [12]).

As far as to the general conformally flat *circular* stationary axisymmetric metric is concerned, Collinson [1] established the following theorem: Every conformally flat stationary axisymmetric spacetime is necessarily static. Strictly speaking, the rigorous statement of the Collinson's theorem should be: Every conformally flat *circular* stationary axisymmetric spacetime is necessarily static. The adjective *circular* should be omitted if the non existence of non-circular spacetimes of the studied classes is proved; in the positive case obviously the use of the adjective *circular* should be redundant.

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